Additivity problems and tensor powers of quantum channels

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Quantum states and channels

- A quantum state ρ (in finite dimension) is a positive semi-definite Hermitian operator with trace one on a Hilbert space $\mathbb{C}^n$.
- A channel can be written as

$$\Phi(\rho) = \mathrm{Tr}_{\mathbb{C}^k}[V\rho V^*]$$

Here, $V : \mathbb{C}^l \to \mathbb{C}^k \otimes \mathbb{C}^n$ is a partial isometry. This means that a channel is completely positive and trace preserving.

Complementary channels

When the input $\rho = |x\rangle\langle x|$ is a rank-one projection the following two matrices share the same non-zero eigenvalues.

$$\mathrm{Tr}_{\mathbb{C}^k}[V\rho V^*] \sim \mathrm{Tr}_{\mathbb{C}^n}[V\rho V^*]$$

Indeed, $V|x\rangle \in \mathbb{C}^k \otimes \mathbb{C}^n$ has the Schmidt decomposition:

$$\sum_i \sqrt{r_i}\,|u_i\rangle \otimes |v_i\rangle$$

where $r_i > 0$ is a probability distribution, and $\{u_i\}, \{v_i\}$ are orthonormal in $\mathbb{C}^k$ and $\mathbb{C}^n$.

We define the complementary channel of Φ by [1]

$$\Phi^c(\rho) = \mathrm{Tr}_{\mathbb{C}^n}[V\rho V^*]$$

[1] [Holevo][King, Matsumoto, Nathanson, Ruskai]

Minimum output entropy (MOE)

The minimal output entropy of channel Φ is defined by

$$S_{\min}(\Phi) = \min_{\rho} S(\Phi(\rho))$$

where ρ are input states. [King, Ruskai]

Here, the von Neumann entropy $S(\cdot)$ of quantum state ρ is:

$$S(\rho) = -\operatorname{Tr}[\rho \log \rho] = -\sum_{i=1}^{d} \lambda_i \log \lambda_i$$

where λ_i are eigenvalues of ρ. Note that

$$S(\rho \otimes \sigma) = S(\rho) + S(\sigma)$$

Holevo capacity (HC)

Holevo capacity of channel Φ is defined as:

$$\chi(\Phi) = \max_{p_i,\rho_i} \left[S(\Phi(\sum_i p_i \rho_i)) - \sum_i p_i S(\Phi(\rho_i)) \right]$$

where $\{p_i, \rho_i\}$ is an ensemble. [Holevo][Schumacher, Westmoreland]

We have an easy bound: $\chi(\Phi) \leq \log d - S_{\min}(\Phi)$

The above bound is saturated when, for example,

$$\Phi(U_g \rho U_g^*) = U_g \Phi(\rho) U_g^*$$

where $g \mapsto U_g \cdot U_g^*$ is an irreducible representation. [Holevo]

Remarks on MOE and HC

- MOE measures purity of channels by considering optimal output while HC is connected to the capacity $C(\cdot)$:

$$C(\Phi) = \lim_{n\to\infty} \frac{1}{n}\chi(\Phi^{\otimes n})$$

  Without entangled inputs or if additivity of χ holds, then

$$C(\Phi) = \chi(\Phi)$$

- Since von Neumann entropy is concave, MOE is achieved by pure input states. This means,

$$S_{\min}(\Phi) = S_{\min}(\Phi^c)$$

- To calculate HC, we need to know about more than just one output state, and in general

$$\chi(\Phi) \neq \chi(\Phi^c)$$

Additivity violation

Write quantum channels:

$$\Phi(\rho) = \mathrm{Tr}_{\mathbb{C}^k}\left[V\rho V^*\right]$$

and their complex conjugate channels:

$$\bar{\Phi}(\rho) = \mathrm{Tr}_{\mathbb{C}^k}\left[\bar{V}\rho V^T\right]$$

Then, with high probability we have additivity violation [2] :

$$S_{\min}(\Phi \otimes \bar{\Phi}) < S_{\min}(\Phi) + S_{\min}(\bar{\Phi})$$

Note that, for any channels Φ and Ω,

$$\min_{\rho\otimes\sigma} S((\Phi \otimes \Omega)(\rho \otimes \sigma)) = \min_{\rho} S(\Phi(\rho)) + \min_{\sigma} S(\Omega(\sigma))$$

[2] [Hastings]: more precisely, another model was used.

Hastings proved:

$$S_{\min}(\Phi) + S_{\min}(\bar{\Phi}) - S_{\min}(\Phi \otimes \bar{\Phi}) \sim \frac{\log k}{k}$$

by using a "random random unitary channel" with $1 \ll k \ll n$:

$$\Phi(\rho) = \sum_{i=1}^{k} r_i U_i \rho U_i^*$$

where

- $U_i \in \mathcal{U}(n)$ are i. i. d.
- $r_i \sim \sum_{j=2n(i-1)+2}^{2ni} X_j^2 \Big/ \sum_{j=1}^{2nk} X_i^j$
  where X_i are i. i. d. normal distributions.

Entangled inputs can improve the capacity - sketchy

1. We know that there is a channel such that

$$S_{\min}(\Phi \otimes \bar{\Phi}) = S_{\min}(\Phi) + S_{\min}(\bar{\Phi}) - \epsilon \qquad \text{where} \quad \epsilon > 0$$

2. This implies [3] that $\Omega = \Phi \oplus \bar{\Phi}$ gives

$$S_{\min}(\Omega^{\otimes 2}) = 2S_{\min}(\Omega) - \epsilon$$

   Then, there exists [4] a channel Ψ such that

$$\chi(\Psi \otimes \Psi) \geq 2\chi(\Psi) + \epsilon$$

3. So, we have

$$C(\Psi) = \lim_{n\to\infty} \frac{1}{2n} \cdot \chi\left(\Psi^{\otimes 2n}\right) \geq \lim_{n\to\infty} \frac{1}{2} \cdot \chi\left(\Psi^{\otimes 2}\right) = \chi(\Psi) + \frac{\epsilon}{2}$$

   I.e.,entangled inputs improve the classical capacity: $C(\cdot)$.

[3] [Fukuda, Wolf]
[4] [Shor]

Additivity question for regularized quantities

- Classical capacity:

$$C(\Phi \otimes \Omega) \stackrel{?}{=} C(\Phi) + C(\Omega) \qquad \text{for } \Phi \neq \Omega$$

- Regularized minimum output entropy:

$$\hat{S}_{\min}(\Phi \otimes \Omega) \stackrel{?}{=} \hat{S}_{\min}(\Phi) + \hat{S}_{\min}(\Omega) \qquad \text{for } \Phi \neq \Omega$$

Here,

$$\hat{S}_{\min}(\Phi) = \lim_{N \to \infty} \frac{1}{N} \cdot S_{\min}(\Phi^{\otimes N})$$

Remark: Non-additivity of $\hat{S}_{\min}(\cdot)$ implies non-additivity of $C(\cdot)$.

Finding counterexamples

Concrete counterexamples for $1 \leq p \leq 2$ are still open.

Remark:

- Concrete counterexamples for the following additivity violation were found [Grudka, M. Horodecki,Pankowski]:

$$S_{p,\min}(\Phi \otimes \Phi) < S_{p,\min}(\Phi) + S_{p,\min}(\Phi) \qquad p > 2$$

  Here,

$$S_{p,\min}(\Phi) = \min_{\rho} S_p(\Phi(\rho))$$

  where S_p is the Renyi p-entropy: $S_p(\sigma) = \frac{p}{1-p} \log \|\sigma\|_p$.

- Irreducible subspaces of group representations are being investigated by Brannan and Collins.

Tensor of "conjugate pair" has rather small entropy

Suppose we have a quantum channel

$$\Phi(\rho) = \mathrm{Tr}_{\mathbb{C}^n}\left[V\rho V^*\right]$$

where

$$V : \mathbb{C}^l \to \mathbb{C}^n \otimes \mathbb{C}^k$$

is an isometry. Then, for $|b\rangle$ a Bell state,

$$\langle b_k| \left[\Phi \otimes \bar{\Phi}(|b_l\rangle\langle b_l|)\right] |b_k\rangle \geq \frac{l}{kn}$$

This means that $\Phi \otimes \bar{\Phi}$ has an output with a large eigenvalue. Additivity violation for $1 < p \leq \infty$ was shown via this trick [5], and for $p = 1$ later.

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[5][Hayden, Winter]

Tensor of conjugate pair - Example

The idea behind is:

$$U \otimes \bar{U} \, |b_m\rangle = |b_m\rangle$$

for $U \in \mathcal{U}(m)$.

For example, take a random unitary channel:

$$\Psi(\rho) = \frac{1}{k} \sum_{i=1}^{k} U_i \rho U_i^*$$

so that

$$\Psi \otimes \bar{\Psi}(|b\rangle\langle b|) = \frac{1}{k}|b\rangle\langle b| + \frac{1}{k^2} \sum_{i \neq j} (U_i \otimes \bar{U}_j) \, |b\rangle\langle b| \, (U_i^* \otimes U_j^T)$$

Single channel has rather large entropy

What are typical outputs for randomly selected channels like?

$$|a\rangle\langle a| \mapsto V|a\rangle\langle a|V^* = |w\rangle\langle w| \mapsto \mathrm{Tr}_{\mathbb{C}^n}\left[|w\rangle\langle w|\right] = WW^*$$

- $|a\rangle$ is a fixed vector in $\mathbb{C}^l$.
- $V|a\rangle$ is a random vector in $\mathbb{C}^k \otimes \mathbb{C}^n$.
- WW^* is the normalized Wishart matrix.

The probability density of WW^* is proportional to:

$$\delta\left(1 - \sum_{1\le i\le k} p_i\right) \prod_{1\le i<j\le k} (p_i - p_j)^2 \prod_{1\le i\le k} p_i^{n-k}$$

The last factor shows that $n \gg k$ implies concentration of eigenvalues. So, typical outputs have rather large entropy.

Aubrun-Szarek-Werner approach for $p > 1$

Define a random quantum channel Φ by the random isometry:

$$V : \mathbb{C}^{n^{1+1/p}} \to \mathbb{C}^n \otimes \mathbb{C}^n.$$

First, $\Phi \otimes \bar{\Phi}$ has a large output eigenvalue larger than $n^{-1+1/p}$.

Second, for a fixed input ρ

$$\text{Typically } \|\Phi(\rho)\|_\infty \sim n^{-1} \quad \text{and} \quad \mathbb{E}\,\|\Phi(\rho)\|_p \sim n^{-1+1/p}.$$

However, by Dvoretzky's theorem we can show that typically

$$\max_\rho \|\Phi(\rho)\|_p \sim n^{-1+1/p} \leq \max_{\hat{\rho}} \|\Phi \otimes \bar{\Phi}(\hat{\rho})\|_p.$$

Third, this translates into, as $n \to \infty$,

$$S_{p,\min}(\Phi) \sim S_{p,\min}(\Phi \otimes \bar{\Phi}).$$

Of course then we have violation for large n.

$$S_{p,\min}(\Phi) + S_{p,\min}(\bar{\Phi}) > S_{p,\min}(\Phi \otimes \bar{\Phi}).$$

What are candidates for optimal inputs for $\Phi \otimes \bar{\Phi}$?[6]

Take a random quantum channels defined by

$$\Phi_n(\rho) = \mathrm{Tr}_{\mathbb{C}^n}\left[V \rho V^*\right]$$

with

$$V : \mathbb{C}^l \to \mathbb{C}^{kn}$$

where $l = tkn$, $k \in \mathbb{N}$, $t \in (0, 1)$ are fixed and $n \to \infty$.

Then, we investigated the asymptotic behavior (as $n \to \infty$) of output eigenvalues of

$$Z_n = \Phi_n \otimes \bar{\Phi}_n(|a_n\rangle\langle a_n|)$$

where $(a_n)_{n \in \mathbb{N}}$ is a fixed sequence of unit vectors.

[6] [Collins, F, Nechita]

We found that the empirical eigenvalue distribution of the matrix Z_n converges *almost surely*, as $n \to \infty$, to:

$$\frac{1}{k^2}\left[\delta_{\lambda_1} + (k^2-1)\delta_{\lambda_2}\right] \, dx$$

where the Dirac masses are located at

$$\lambda_1 = t|m|^2 + \frac{1-t|m|^2}{k^2} \quad \text{and} \qquad \lambda_2 = \frac{1-t|m|^2}{k^2}.$$

if

$$\frac{\mathrm{Tr}\left[A_n\right]}{\sqrt{l}} = m + O\left(\frac{1}{n^2}\right)$$

Here, $|a_n\rangle \leftrightarrow A_n$ is the correspondence $\mathbb{C}^l \otimes \mathbb{C}^l \leftrightarrow M_l(\mathbb{C})$.

Conclusion: The Bell state is best. Examine, for example,

$$a = \sum_i \alpha_i \, |i\rangle \otimes |i\rangle$$

Remark. Maximally mixed state for $\Phi \otimes \Phi$, $\Phi \otimes \Phi^T$ or $\Phi \otimes \Phi^*$.

How about tensor powers $(\Phi \otimes \bar{\Phi})^{\otimes r}$?[7]

Our calculation shows that tensor-products of Bell states are best. Suppose we have a random quantum channel:

$$\overset{1}{\Phi} \otimes \overset{2}{\Phi} \otimes \cdots \otimes \overset{r}{\Phi} \otimes \overline{\overset{\hat{1}}{\Phi}} \otimes \overline{\overset{\hat{2}}{\Phi}} \otimes \cdots \otimes \overline{\overset{\hat{r}}{\Phi}}$$

where best inputs are

$$|b_{\pi(1),\hat{1}}\rangle \otimes |b_{\pi(2),\hat{2}}\rangle \otimes \cdots \otimes |b_{\pi(r),\hat{r}}\rangle$$

where $\pi \in S_r$. Here, $|b_{i,j}\rangle$ is a Bell state over the i-th space for Φ and j-th space for $\bar{\Phi}$.

Remark. Hastings conjectured that violation of additivity happens only within each conjugate pair.

[7] [F, Nechita]

How about tensor powers $\Phi^{\otimes 2r}$, where Φ is orthogonal ? [8]

This time, we generate random channels by orthogonal matrices instead of unitary ones. So, $\bar{\Phi} = \Phi$.

$$\overset{1}{\Phi} \otimes \overset{2}{\Phi} \otimes \cdots \otimes \overset{r}{\Phi} \otimes \overset{r+1}{\Phi} \otimes \overset{r+2}{\Phi} \otimes \cdots \otimes \overset{2r}{\Phi}$$

where best inputs are

$$\bigotimes_{c \in \pi} |b_c\rangle$$

where π is a paring of $2r$ elements. Here, $|b_c\rangle$ is a Bell state over the i-th and j-th spaces when $c = (i, j)$.

We conjecture that typically for orthogonal case

$$S_{\min}(\Phi^{\otimes 2r}) = r\, S_{\min}(\Phi^{\otimes 2})$$

or, we can make it weaker:

$$\lim_{r \to \infty} \frac{1}{r} S_{\min}(\Phi^{\otimes r}) = \frac{1}{2} S_{\min}(\Phi^{\otimes 2})$$

[8] [F, Nechita]

Montanaro's multiplicative bound

$$\|\Phi^{\otimes r}\|_{1\to\infty} \leq (\|VV^*\|_\infty)^r$$

where V is the isometry defining Φ.

F-Nechita's multiplicative bound

$$\|\Phi^{\otimes r}\|_{1\to 2} \leq \left(\|C_\Phi^\Gamma\|_\infty\right)^r$$

where C_Φ^Γ is the partially transposed Choi matrix of Φ.

Then the bounds lead to the following weak additivity respectively for $p=\infty, 2$: typically under random choice of channels

$$S_{p,\min}(\Phi^{\otimes r}) \geq \frac{r}{2} S_{p,\min}(\Phi)$$

Montanaro first described it as "weakly multiplicative", in terms of maximum output p-norms.

F-Gour's multiplicative bound

For a unital quantum channel: $M_n(\mathbb{C}) \to M_k(\mathbb{C})$,

$$\|\Phi^{\otimes r}\|_{1\to 2} \leq (\gamma_\Phi)^{r/2}\,.$$

Here,

$$\gamma_\Phi = \frac{1}{k} + \left(1 - \frac{1}{n}\right) \|D_\Phi D_\Phi^*\|_\infty$$

where D_Φ is the dynamical matrix of Φ restricted on trace-less Hermitian matrices.

We also got an upper bound for the classical capacity:

$$C(\Phi) \leq \log k + \log \gamma_\Phi.$$

This bound is saturated by the Werner-Holevo channel.

Summary

- Additivity violation may be a special phenomena for conjugate pairs.
- Perhaps, additivity violation typically does not hold for $\Phi^{\otimes n}$ when Φ is generated by unitary group.
- Otherwise, we need to know how fast non-additivity grows and how much contribution it makes for regularized quantity.

Thank you very much for your patience.