Spectral convergence of Riemannian vector bundles

Atsushi KASUE (Kanazawa University)

A spectral distance on a set of compact Riemannian manifolds is introduced by means of their heat kernels, and it is proved that a family of compact Riemannian manifolds whose heat kernels uniformly satisfy on-diagonal upper estimates is precompact and further any Cauchy sequence in the family converges to a compact regular Dirichlet space in such a way that the eigenvalues and eigenfunctions of the manifolds tend to those of the limit space([1],[2]). The uniform topology induced from the spectral distance is related to that of the Gromov-Hausdorff distance and also the topology of Mosco-convergence of energy forms introduced by Kuwae and Shioya [3].

In this talk, we are concerned with energy forms of Hermitian vector bundles endowed with metric connections over compact Riemannian manifolds whose heat kernels uniformly satisfy on-diagonal upper estimates.

The main result is stated in the following

**Theorem 0.1** Let \( \{E_n \to B_n\} \) be a sequence of Hermitian vector bundles of the same rank \( r \) endowed with metric connections over compact Riemannian manifolds \( B_n \). Suppose that the heat kernels \( p_{B_n} \) of \( B_n \) satisfy \( p_{B_n}(t,x,y) \leq A/t^{r/2} \) for some positive constants \( A \) and \( r \), and for all \( t \in (0,1] \) and \( x,y \in B_n \). Then there exists a subsequence \( \{E_k\} \) of \( \{E_n\} \) and a (symmetric) closed form \( (\mathcal{F},D[\mathcal{F}]) \) on a Hilbert space to which the energy forms on the Hilbert space of \( L^2 \)-sections of \( E_k \) Mosco-converges as \( k \to \infty \). Moreover if \( r = 1 \), then the Hilbert space consists of \( L^2 \)-sections of a continuous Hermitian line bundle over an open subset in a compact metric space with a Radon measure.
Here we sketch the proof of the main theorem. Let $\{E_n \to B_n\}$ be as in the theorem. Let $M_n$ be the principal $U(r)$-bundle of unitary frames of $E_n$. Then passing to a subsequence, we see that $M_n$ Mosco-converges, as $n \to \infty$, to a compact Dirichlet space on which the unitary group keeps to act continuously; the measure and the Dirichlet form are invariant under the action. The limit Hilbert space and the closed form in the main theorem are respectively given by the $L^2$-closure of the space of continuous functions of the Dirichlet space to $C^r$ which are equivariant under the action of $U(r)$ and the form restricted to this space. If the vector bundles are of rank one, then the outside of the subset of points at which all equivariant continuous functions to $C$ vanish is invariant under the action of $U(1)$ and we get a continuous Hermitian line bundle over the quotient space which is associated with the canonical action of $U(1)$ on $C$. This is the limit bundle mentioned in the main theorem.

The topic taken in this talk goes back to the note of the same title which was written in Japanese and published in Lecture Note Series in Mathematics 7, Osaka University, 2002.

References

