Ricci curvature and $L^p$-convergence

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A main goal in this talk is to introduce the notion of the $L^p$-convergence of tensor fields with respect to the Gromov-Hausdorff topology given in [1]. The precise statement is as follows:

Let $n$ be a positive integer, $K$ a real number and let $(M_i, m_i, \text{vol})_{i \in \mathbb{N}}$ be a sequence of renormalized pointed complete $n$-dimensional Riemannian manifolds with $\text{Ric}_{M_i} \geq K(n-1)$ and $M_i \neq \{m_i\}$, where $\text{vol} := \text{vol}/\text{vol}\ B_1(m_i)$.

Cheeger-Colding showed that the cotangent bundle $\pi^*_1 : T^* M_i \to M_i$ of $M_i$ exists in some sense. It is a fundamental property of the cotangent bundle that every Lipschitz function $f$ on a Borel subset $A$ of $M_i$ has the canonical section $d f(x) \in T^*_x M_i$ for a.e. $x \in A$. We also define the tangent bundle $\pi^*_0 : T M_i \to M_i$ of $M_i$ by the dual vector bundle of $T^* M_i$ and denote the dual section of $d f : A \to T M_i$. We will denote by $\langle \cdot, \cdot \rangle$ the canonical metric on $T^*_r s M_i$ and by $L^p(T^*_r s A)$ the space of $L^p$-sections of $T^*_r s M_i$ over $A$.

Let $r, s \in \mathbb{Z}_{\geq 0}$, $R > 0$, $1 < p < \infty$ and $T_i \in L^p(T^*_r B_R(m_i))$ for every $i \leq \infty$ with $\sup_{i \leq \infty} ||T_i||_{L^p} < \infty$, where $B_R(m_i) := \{x_i \in M_i; x_i - m_i < R\}$ and $x_i, m_i$ is the distance between $x_i$ and $m_i$.

We now consider the following question:

**Question:** Can we define the following?

**W** $T_i \ \text{$L^p$-converges weakly to $T_\infty$.}$

**S** $T_i \ \text{$L^p$-converges strongly to $T_\infty$.}$

A difficulty to give the definitions above is that we can NOT consider the difference `$T_i - T_\infty$' canonically because it would be hard to compare between $T^*_r s M_i$ and $T^*_r s M_\infty$.

A main purpose in this talk is to answer the question above, it is YES:

**Definition 0.1 ([1]).**
We say that $T_i L^p$-converges weakly to $T_\infty$ on $B_R(m_\infty)$ if for every $x_\infty \in B_R(m_\infty)$, every $\{z_i\}_{1 \leq i \leq r+s} \subset M_\infty$ and every $r > 0$ with $B_r(x_\infty) \subset B_R(m_\infty)$ we have

$$\lim_{j \to \infty} \int_{B_r(x_j)} \left\langle T_j, \bigotimes_{i=1}^{r} \nabla r_{z_{i,j}} \bigotimes_{i=r+1}^{r+s} dr_{z_{i,j}} \right\rangle d\text{vol} = \int_{B_r(x_\infty)} \left\langle T_\infty, \bigotimes_{i=1}^{r} \nabla r_{z_{i}} \bigotimes_{i=r+1}^{r+s} dr_{z_{i}} \right\rangle dv,$$

where $x_j \to x_\infty$, $z_{i,j} \to z_i$ as $j \to \infty$ and $r_z$ is the distance function from $z$.

We say that $T_i L^p$-converges strongly to $T_\infty$ on $B_R(m_\infty)$ if $T_i L^p$-converges weakly to $T_\infty$ on $B_R(m_\infty)$ and $\limsup_{i \to \infty} \|T_i\|_{L^p(B_R(m_i))} \leq \|T_\infty\|_{L^p(B_R(m_\infty))}$.

The fundamental properties of the $L^p$-convergence above have several applications. Roughly speaking, their applications include the following:

1. A Rellich type compactness result with respect to the GH-topology.
2. A Bochner type inequality on $M_\infty$.
3. The $L^2$-weak convergence of the Hessians of most functions with respect to the GH-topology.
4. The continuity of the first eigenvalues of the $p$-Laplacian with respect to the GH-topology.
5. The lower semicontinuity of the dimensions with respect to the GH-topology.

I would like to explain these results as long as time allows.

References